# Classification of quantum critical states of integrable antiferromagnetic spin chains and their correspondent two-dimensional topological phases

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We examine the effective field theory of the Bethe ansatz integrable Heisenberg antiferromagnetic spin chains. It shows that the quantum critical theories for the integer spin-$S$ chains should be characterized by the $SO(3)$ level-$S$ Wess-Zumino-Witten model, and classified by the third cohomology group $H^3(SO(3), Z) = Z$. Depending on the parity of spin $S$, this integer classification is further divided into two distinct universality classes, which are associated with two completely different conformal field theories: the even-$S$ chains have gapless bosonic excitations and the odd-$S$ chains have both bosonic and fermionic excitations. We further show that these two classes of critical states correspond to the boundary states of two distinct topological phases in two dimension, which can be described by two-dimensional doubled $SO(3)$ topological Chern-Simons theory and topological spin theory, respectively.

*Introduction.*- The study of topological phases and their classification has become an important issue in condensed matter physics. Historically, topological order[1] was proposed to describe fractional quantum Hall states,[2, 3] which cannot be characterized by a local order parameter with spontaneous symmetry breaking. The nature of these states is long-range entanglement and a finite degeneracy of the ground states on a torus.[4, 5] Recently, it is shown that some nontrivial properties, such as robust gapless edge excitations, exist even in quantum gapped states with only short-ranged entanglement, known as symmetry protected topological (SPT) phases. The best example is the state of topological insulators[6, 7] protected by time reversal symmetry and charge conservation symmetry. Actually, such SPT phases also exist in one-dimensional quantum antiferromagnetic spin chains.[5]

More than thirty years ago, from the theory of nonlinear sigma model with $\theta$ term, Haldane[8] predicted that antiferromagnetic Heisenberg spin chains are classified into two universality classes: half-odd integer spins with gapless excitations and integer spins with gapped excitations. Recent studies[9, 10] indicated that the Haldane gapped phase for an odd integer spin chain is a nontrivial SPT phase, while the even integer spin chain is trivial, because the edge states are not symmetry protected. The number of SO(3) symmetry protected distinct topological phases can be labeled by the elements of the second group cohomology of $SO(3)$ group[11, 12]: $\mathcal{H}^2(SO(3), U(1)) = Z_2$. In order to find a complete classification scheme including the critical states of the half-odd-integer spin antiferromagnetic chains, it is desirable to consider the problem as whether the differences between the odd and even integer Haldane gapped phases can be understood from their effective field theo-

ries. Moreover the classification of one-dimensional quantum critical states is of itself an important issue, because of its close relation to the boundary theories of two-dimensional topological ordered phases.

In this Letter, we will consider this issue using the Bethe ansatz integrable antiferromagnetic spin chains with arbitrary spin.[13, 14] It has been known that the critical states for the half-odd integer spins are described by the $SU(2)$ level-$2S$ Wess-Zumino-Witten (WZW) model.[15–18] Realizing that it is the $SO(3)$ group symmetry as the faithful representation for integer spins, we point out that the quantum critical states for the integer spin chains should be regarded as the $SO(3)$ level-$S$ WZW models, which correspond to a two-dimensional (doubled) $SO(3)$ topological spin gauge theories[19]. Depending on the parity of spins, there exist two universality classes with completely different conformal field theories (CFTs): the even integer chains have bosonic excitations and the odd integer chains have both bosonic and fermionic excitations. Such a classification is consistent with the classification for the 1D $SO(3)$ gapped phases from the second group cohomology,[11, 12] and is further shown to have an intimate relation to 2D $SO(3)$ SPT phases or chiral spin liquid phases.

*Model Hamiltonian.*- We begin with the $SU(2)$ quantum spin chains, where the spin operators on each site take values in the spin $su(2)$ algebra. The relevant irreducible representations are labeled by the spin $S \in \{1/2, 1, 3/2, ...\}$, including the half-odd integer and integer spins. The standard one-dimensional spin-$S$ antiferromagnetic Heisenberg model is given by

$$H = J \sum_{n=1}^{L} \mathbf{S}_n \cdot \mathbf{S}_{n+1}, \ S_n^2 = S(S+1), \ \ J > 0. \quad (1)$$

For the $S = 1/2$ case, a quantum critical phase was ob-

tained by the Bethe ansatz solution, and spin-spin correlation functions show power law behavior. However, no exact solution exists for $S > 1/2$.

For the $S = 1$ case, a general $SU(2)$ symmetric model has the form

$$H = \sum_{n=1}^{L} \left[ (\cos\theta)\, \mathbf{S}_n \cdot \mathbf{S}_{n+1} + (\sin\theta)\, (\mathbf{S}_n \cdot \mathbf{S}_{n+1})^2 \right], \quad (2)$$

where the Haldane gapped phase has been confirmed in the parameter range $-\pi/4 < \theta < \pi/4$, and the dimerized phase is for $-3\pi/4 < \theta < -\pi/4$. There is a quantum critical point at $\theta = -\pi/4$, which is also exactly solved by the Bethe ansatz method.[13, 14] We can understand the nature of both the Haldane gapped phase and dimerized phase from the critical field theory, which has been characterized by the $SU(2)$ level-2 or $SO(3)$ level-1 WZW model in term of three free Majorana fermions. Depending on the sign of the relevant mass term, an energy gap is generated for the Haldane phase and dimerized phase, respectively.[20]

Actually, there is a generalization of the antiferromagnetic Heisenberg spin chain to arbitrary spin-$S$ with preserving the $SU(2)$ symmetry and integrability. The model Hamiltonian can be defined by[13, 14]

$$H_S = J \sum_{n=1}^{L} Q_{2S}(\mathbf{S}_n \cdot \mathbf{S}_{n+1}), \quad (3)$$

where $S_n = (S_n^x, S_n^y, S_n^z)$ is the $SU(2)$ generator of arbitrary integer or half-odd integer spin, and the special polynomial of degree $2S$ is

$$Q_{2S}(X) = -\sum_{j=1}^{2S} \left( \sum_{k=1}^{j} \frac{1}{k} \right) \prod_{l=0, l \neq j}^{2S} \left( \frac{X - X_l}{X_j - X_l} \right)$$

with $X_j = \frac{1}{2}[j(j+1) - 2S(S+1)]$. For $S = 1/2$, the Hamiltonian $H_{1/2}$ is the antiferromagnetic Heisenberg spin model (1), while the Hamiltonian for $S = 1$ corresponds to the model (2) at the quantum critical point $\theta = -\pi/4$.

*Critical field theories.-* Generally, one can expect that the Bethe ansatz integrable Hamiltonians with arbitrary spin describe a family of quantum critical states with the $SU(2)$ symmetry, which can be described by (1+1)D CFTs. When the conformal central charge $c = 3S/(S+1)$ and thermodynamic properties are compared with the Bethe ansatz results for several small spin cases,[15, 17, 18] the effective critical field theory had been suggested by the $SU(2)$ level-2S WZW model with the action:

$$\begin{aligned} S(g) &= -\frac{ik}{12\pi} \int d^3 y \epsilon^{\mu\nu\lambda} \mathrm{Tr}\left[ g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\lambda g \right] \\ &+ \frac{k}{8\pi} \int dx d\tau \, \mathrm{Tr}\left( \partial_\mu g \partial_\mu g^{-1} \right), \end{aligned} \quad (4)$$

where $g \in SU(2)$ is the group element and $k = 2S$ is the level index. When a closed boundary condition is imposed, the first term is the (1+1)D WZW action. Actually, the above (1+1)D WZW model can be considered as the boundary theory of the following (2+1)D principal chiral model[21]

$$\begin{aligned} S_{\mathrm{PC}} &= \int d^2 x d\tau \left[ \frac{1}{4\kappa^2} \mathrm{Tr}(\partial_\mu g \partial_\mu g^{-1}) \right. \\ &\left. -\frac{ik}{12\pi} \varepsilon^{\mu\nu\lambda} \mathrm{Tr}(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\lambda g) \right] \end{aligned} \quad (5)$$

with the coupling constant $\kappa^2 \to \infty$ in the strong coupling limit, where the second term is quantized on the closed space-time manifold. When the space is open, the second term becomes an effective WZW term for the boundary theory and flows to the fixed point (4).

However, there exist subtleties related to the symmetry group, which has been neglected in the previous critical effective field analysis.[15–18] When we lift the symmetry described in terms of the $su(2)$ Lie algebra to a group symmetry of $SU(2)$, we might have several choices, and not all of them will lead to an $SU(2)$ faithful representation.[12] If the physical spins transform in half-odd integer spin representations, the group $SU(2)$ is acting faithfully. However, if the physical spins transform in integer spin representations, the $SU(2)$ does not act faithfully and the actual symmetry should be $SO(3) = SU(2)/Z_2$. The $SU(2)$ is a two-fold covering of the $SO(3)$. Moreover, it is important to understand the differences between $SU(2)$ and $SO(3)$ more precisely. When viewed as geometric manifolds, $SU(2)$ and $SO(3)$ look *locally* identical. However, they differ in their *global* topology: $SU(2)$ is simply-connected and the first homotopy group $\pi_1(SU(2)) = 0$; while $SO(3)$ is non-simply connected, $\pi_1(SO(3)) = Z_2$, namely, it admits non-trivial loops which cannot be contracted to a point.

Therefore, for the integrable half-odd integer spin chains, it is correct to identify the effective field theory as the $SU(2)$ level-2S WZW model. However, for the quantum critical states of the integer spin chains, although the effective action has the same conformal central charge, the corresponding effective field theory should be regarded as the $SO(3)$ level-S WZW model. The corresponding WZW action is given by

$$\begin{aligned} S'(g) &= -\frac{iS}{24\pi} \int d^3 y \epsilon^{\mu\nu\lambda} \mathrm{Tr}\left[ g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\lambda g \right] \\ &+ \frac{S}{16\pi} \int dx d\tau \, \mathrm{Tr}\left( \partial_\mu g \partial_\mu g^{-1} \right), \end{aligned} \quad (6)$$

where $g \in SO(3)$ as the group elements. Similarly, the above (1+1)D WZW model can also be considered as the boundary theory of the following (2+1)D $SO(3)$ principal

TABLE I: Conformal field theory spectra of the $SU(2)_{2S}$ and $SO(3)_S$ WZW models. The central charge of both models is given by $c = 3S/(S+1)$.

| WZW model | | Primary fields $(j, \bar{j})$ |
|---|---|---|
| $SU(2)_{2S}$ | | $(0,0)$, $(\frac{1}{2},\frac{1}{2})$,$(1,1)$,...,$(S,S)$ |
| $SO(3)_S$ | untwisted: | $(0,0)$, $(1,1)$,...,$(S,S)$ |
| ($S$=even) | twisted: | $(0,S)$, $(1, S-1)$,...,$(S,0)$ |
| $SO(3)_S$ | untwisted: | $(0,0)$, $(1,1)$,...,$(S,S)$ |
| ($S$=odd) | twisted: | $(\frac{1}{2}, S - \frac{1}{2})$, $(\frac{3}{2}, S - \frac{3}{2})$,...,$(S - \frac{1}{2}, \frac{1}{2})$ |

chiral model

$$
\begin{aligned}
S'_{\text{PC}} &= \int d^2\mathbf{x}d\tau \left[ \frac{1}{8\kappa^2}\text{Tr}(\partial_\mu g \partial_\mu g^{-1}) \right. \\
&\left. - \frac{iS}{24\pi}\varepsilon^{\mu\nu\lambda}\text{Tr}(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\lambda g) \right], \quad (7)
\end{aligned}
$$

which has a global $SO(3)_L \times SO(3)_R$ symmetry.[22] Namely, the action $S'_{\text{PC}}$ is invariant under the transformation $g(\tau, \mathbf{x}) = \Omega g(\tau, \mathbf{x})\bar{\Omega}^{-1}$, where $\Omega \in SO(3)_L$ and $\bar{\Omega} \in SO(3)_R$ are independent constant group elements. However, in the boundary theory of (6), an enhanced local symmetry exists $g(\tau, x) \rightarrow \Omega(\tau + x)g(\tau, x)\bar{\Omega}^{-1}(\tau - x)$.

*Excitation spectra from CFT.-* So far the obtained results for integer spin $S$ seem to indicate that the $SO(3)$ level-$S$ WZW model are refinements of the $SU(2)$ level-$2S$ WZW model. Actually, the critical field theories with odd and even integer spins represent two distinct universality classes of the quantum critical states, which are completely different from those quantum critical theories of the $SU(2)$ level-$2S$ WZW model. By performing modular invariance to the $SO(3)$ level-$S$ WZW model[23] or constructing the space of states in the canonical quantized theory[24], the CFT spectra have been worked out in terms of the primary fields $(j, \bar{j})$, where $j$ and $\bar{j}$ are the left $SO(3)_L$ and right $SO(3)_R$ isospin quantum number.

For the $SO(3)$ level-$S$ WZW model with *even* integer $S$, the corresponding CFT includes the primary fields, consisting of the untwisted and twisted sectors with multiplicity one.[23, 24] (see Tab.I) Since all these primary fields have integer left and right isospin quantum numbers, they represents *bosonic* fields with integer conformal weights, forming a chiral algebra. The field $(0,0)$ denotes the identity $I$ and some primary fields appearing in both sectors correspond to two different primary fields which have the same transformation properties under the current algebra. The field $(1,1)$ in the untwisted sector is nothing but the $SO(3)$ symmetric elementary field $g(\tau, x)$, and the operator products of $g$ with itself generate all the primary fields in the untwisted sector. The primary fields in the twisted sector are so-called spin fields, which are generated by the elementary soliton field $(0,S)$. There is an important selection (fusion) rule:

$(0,S) \times (0,S) = I$. Actually, the operator products of the elementary and soliton fields with itself or sufficient number of times can give rise to all the primary fields. So they represent elementary fields of the critical field theory.[23, 24]

For the $SO(3)$ level-$S$ WZW model with *odd* integer $S$, the primary fields also consist of the untwisted sector and twisted sector with multiplicity one.[23, 24] (see Tab.I) Since the twisted primary fields contain half-odd integer isospin quantum numbers, the conformal weights of these primary fields include both half-odd-integers (*fermionic*) and integers (*bosonic*), forming a $Z_2$ graded chiral algebra or chiral super-algebra.[19] Moreover, the selection (fusion) rule for the product of the elementary spin field with itself becomes[23]

$$
(\frac{1}{2}, S - \frac{1}{2}) \times (\frac{1}{2}, S - \frac{1}{2}) = I + (1,1), \quad (8)
$$

which indicates that the elementary primary field may be generated from the operator product of the elementary spin field with itself.

Therefore, the $SO(3)$ level-$S$ WZW model is divided into two universality classes: one is denoted by the even level index with chiral algebra and another corresponds to the odd level index with chiral super-algebra. Such a classification is consistent with the analysis of the D-brane charge groups,[25] where the even level theories have a charge group $Z_2 \times Z_2$, while the odd level theories have a charge group $Z_4$.

As a comparison, the CFT spectrum of the $SU(2)$ level-$2S$ WZW model is also given in Tab.I, where a single sector includes the primary fields with both integer and half-odd integer isospin quantum numbers with multiplicity one.[24] However, these primary fields can only form the chiral algebra with integer conformal weights. Interestingly, there is a coincidence for the critical state of the $S = 1$ chain. The obtained CFT primary fields $(0,0)$, $(\frac{1}{2},\frac{1}{2})$, and $(1,1)$ can be given by both the $SO(3)$ level-1 WZW model and the $SU(2)$ level-2 WZW model.

Away from the critical point of the integer spin chains, the elementary spin field is not allowed by the symmetry, and the most relevant operator is given by the elementary field of the WZW model: $\mathcal{L}_{\text{int}} = -\lambda \text{Tr} g$, which corresponds to a mass term of the critical field theory. Such a mass term can produce an energy gap in the spin excitations, leading to the Haldane gapped phase ($\lambda > 0$) or the dimerized phase ($\lambda < 0$), respectively. However, for the $SU(2)$ level-$2S$ WZW model, the above mass operator does not permit, so the leading relevant operator should be given by $\mathcal{L}_{\text{int}} = -\lambda \left( \text{Tr} g \right)^2$ in terms of $g \in SU(2)$.

*Correspondent 2D topological phases.-* According to the work of Dijkgraaf and Witten,[19] the (2+1)D topological Chern-Simons theories with gauge group $G$ are classified by the fourth cohomology group $H^4(BG, Z)$, where $BG$ is the classifying space of the group $G$. For a non-simply connected gauge group $G$, namely, $\pi_1(G) \neq 0$, the level-$k$

can not take arbitrary integer values. However, for the non-simply connected group $G = SO(3)$, $k$ should be an even integer.[19, 26] Notice that our definition of the level $k$ for $SO(3)$ is half of that defined in Ref.[19, 26]. Compare to these references, we should redefine $k \to 2k$. More importantly, the $(2+1)$D $SO(3)$ Chern-Simons topological theory has a correspondence to a $(1+1)$D $SO(3)$ WZW model with even level $k$, and there exists an inverse transgression map: $H^4(BG, Z) \to H^3(G, Z)$.

However, since the $(2+1)$D space-time manifold always support a spin structure, the level $k$ can take more integer values. For $G = SO(3)$, the level-$k$ can be either even or odd integers. The corresponding $SO(3)$ topological gauge theories with an odd level-$k$ are called topological spin theories, and also have the correspondence to the $(1+1)$D $SO(3)$ WZW model with the same level-$k$, which is however associated to a $Z_2$ graded chiral algebra or chiral super-algebra.[19]

Next we will show how the principal chiral model (7) can provide a bridge to connect the $(2+1)$D $SO(3)$ topological spin Chern-Simons theory to the $(1+1)$D $SO(3)$ WZW model with the same level index. To this end, we use the method of "gauging" the symmetry group. Because the model (7) has $SO(3)_L \times SO(3)_R$ symmetry, we can regard this symmetry group as a gauge group and minimally couple to two external gauge fields $A_\mu$ and $\bar{A}_\mu$. Namely we replace $g^{-1}\partial_\mu g$ by $g^{-1}(\partial_\mu + A_\mu)g$ and $g\partial_\mu g^{-1}$ by $g(\partial_\mu + \bar{A}_\mu)g^{-1}$, respectively.[28] After integrating out the group elements, an effective action can be derived

$$S(A, \overline{A}) = \frac{ik}{8\pi} \int_M d^3x \varepsilon^{\mu\nu\lambda} \left[ (A_\mu^a \partial_\nu A_\lambda^a + \frac{i\varepsilon_{abc}}{3} A_\mu^a A_\nu^b A_\lambda^c) - (\bar{A}_\mu^a \partial_\nu \bar{A}_\lambda^a + \frac{i\varepsilon_{abc}}{3} \bar{A}_\mu^a \bar{A}_\nu^b \bar{A}_\lambda^c) \right], \quad (9)$$

which is a doubled Chern-Simons topological gauge theory on a $SO(3)$ spin manifold $M$.[29] Since the two external gauge fields $A_\mu$ and $\bar{A}_\mu$ act independently, it is natural to expect no cross terms appearing in this action.

To justify the self-consistency of our results, we reverse the reasoning of the previous discussion, and derive the $(1+1)$D WZW action from the doubled Chern-Simons action (9). First of all, the action (9) is invariant under time reversal transformation: $i \to -i$, $t \to -t$, $\tau \to \tau$, and $A_\mu \to \bar{A}_\mu$. Following the method of Ref.[26], we can choose a specific gauge and integrate out $A_0$, resulting in a delta function $\delta(F_{ij})$ with $F_{ij} = \partial_i A_j - \partial_j A_i + \frac{1}{2}[A_i, A_j]$. Then substituting the solution $A_i = g\partial_i g^{-1}$ into the first part of (9), we obtain a *chiral* version of a $(1+1)$D $SO(3)$ level-$k$ WZW model,

$$S_1 = \frac{ik}{48\pi} \int_M d^3x \epsilon^{\mu\nu\lambda} \text{Tr}(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\lambda g) + \frac{ik}{16\pi} \int_{\partial M} dx_1 dx_0 \text{Tr}(\partial_1 g^{-1}\partial_0 g), \quad (10)$$

where $x_1$ is the spacial coordinate of the boundary and $x_0 = \tau$. Noticing that $\bar{A}_i$ is the time reversal partner of $A_i$, we can set $\bar{A}_i = g^{-1}\partial_i g$, where we have assumed that this Lagrangian is strictly invariant under time reversal, and then another action $S_2$ with *opposite* chirality can be derived from the second part of (9). Compared to $S_1$, the second term in $S_2$ has a negative sign only, and the first term in $S_2$ is exactly the same. Finally the total action is thus deduced

$$S_{\text{top}} = \frac{ik}{24\pi} \int_M d^3x \epsilon^{\mu\nu\lambda} \text{Tr}(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\lambda g). \quad (11)$$

where the dynamic terms are cancelled and the topological WZW term with time reversal symmetry is left. To get back the dispersion of the WZW model, a dynamic term has to be included

$$S_{\text{dyn}} = \frac{1}{8\kappa^2} \int_{\partial M} dx_1 dx_0 \text{Tr}(\partial_1 g \partial_1 g^{-1} + \partial_0 g \partial_0 g^{-1}). \quad (12)$$

Under the renormalization flow, the fixed point can be reached at $\frac{1}{8\kappa^2} = \frac{k}{16\pi}$, leading to the full action of the $(1+1)$D $SO(3)$ WZW model (6).

Moreover, our doubled Chern-Simons model(9) is essentially a response action. However, if we regard it as the effective action of a dynamic gauge field, it should describe two-dimensional topologically ordered phases with time reversal and parity symmetries, however their boundary excitations are not necessarily gapless.[29] It has been shown that each $(1+1)$D critical field theory of the WZW model corresponds to a $(2+1)$D topologically ordered phase, which can be described by a topological gauge theory or topological spin theory, depending on the parity of level $k$. It should be mentioned that these results are also applicable to the $SU(2)$ level-$k$ WZW models, where there is no distinct difference between even $k$ and odd $k$.

Furthermore, if we regard the symmetry group as one of the chiral symmetry, namely $SO(3)_L$, the resulting effective bulk theory is given by the $SO(3)$ Chern-Simons topological theory.[27] The above classification of the WZW theory can be used to study the $(2+1)$D $SO(3)$ SPT phases, which are classified by the third group cohomology $\mathcal{H}^3(SO(3), U(1)) = Z$, equivalent to the fourth cohomology group $H^4(BSO(3), Z)$. However, from the inverse transgression map, these SPT phases can be described by the principal chiral model (7), which is classified by the third cohomology group $H^3(SO(3), Z)$ with *even* level $k$. These bulk SPT phases are gapped, while the boundary theory has gapless excitations as long as the $SO(3)$ symmetry is preserved. However, for the models with an odd level-$k$, the bulk excitations are still gapped, but we can have a $Z_2$ vortex excitation carrying half-integer spin according to the $SO(3)_L$ symmetry.[27] This has also been reflected in the boundary theory, which contains both integer and half-odd integer excitations. The emergence of fractionalized excitations indicates that the

odd level-$k$ SPT phases represent 2D topologically ordered states with possible nontrivial statistics[30], for instance, the chiral spin liquids. Therefore, the classification of the (1+1)D $SO(3)$ WZW models as the boundary theory of the two-dimensional principal chiral models leads to the above important prediction.

*Conclusion.*- We have carefully examined the effective field theory of the Bethe ansatz integrable Heisenberg antiferromagnetic spin chains. The quantum critical integer spin chains should be characterized by the $SO(3)$ level-$S$ WZW model and divided into two distinct universality classes, determined by the parity of the spin. These two classes of WZW models correspond to 2D doubled $SO(3)$ topological Chern-Simons theory or doubled $SO(3)$ topological spin theory, respectively. Furthermore, if we adopt the chiral symmetry $SO(3)_L$, these two classes of WZW theory describe the boundary excitations of 2D SPT phases or 2D chiral spin liquid phases, respectively. Therefore, our present work provides a systematic method to study 2D topological phases from 1D quantum critical theory. Some other issues, such as the correspondent chiral spin liquid, are under investigations.

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